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C.M.1984/ D:7
Statistics Committee
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A New Method for Smoothing Length Distributions of Fish and for Sharpening Peaks due to Different Year Classes.

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ABSTRACT

If it may be assumed that a length distribution consists of a sum of normal-distributions, one for each year class, then deconvoluting the length distribution with a normal-distribution having a variance v, has the effect of making the variances of all the year classes smaller by the amount v.
- Thus it is possible in one operation to reduce the overlapping between year classes.

In the real world the length distribution of a given species is not known but only a finite sample from it. Instead of just grouping the measurements into some intervals, we propose a different method based on filtering, where we try to eliminate high frequency fluctuation in the measured length ditribution that cannot be told from random variations.

It is shown through a number of examples that these methods are applicable to real data, (length ditributions for cod, haddock and shrimp) collected by the Marine Research Institute of Iceland.

These methods are computationally not difficult and can be fully automated.

INTRODUCTION

It is well known that the size distributions of fish have peaks due to different year classes, ref (3). These peaks are very pronounced for the younger fish, but as the fish grows older the overlapping between the year classes increases. This is mainly due to reduced growth of fish with age but also because of increase in variance in size (length) of each year class.
Methods that take advantage of this fact have long existed for

deriving information on mean lengths and proportions of different year classes from size distributions only (the Petersen's method ref (3), the deviation method ref (6),(5)).

Since otolithic age determinations became common, those methods have decreased in importance and are considered by many to be obsolete or at best old-fashioned curiosities.

It is the belief of the author of this paper that by the near extinction of these methods in fishery sciences much has been lost, especially in that critical graphical presentation of data has given way to mean values of lengths and proportions that pop out of "age-length computer programs". These programs work well enough if the assumptions that they are based on hold, which typically they do not, which can often be seen if one inspects the length distributions.

The methods that we are about to present are not intended to replace otolithic age-determination but rather to supplement it to get an immediate idea of what information a given length distribution contains, to increase confidence in calculated values, and of course if no otolithic age-determinations can be obtained they can be of aid in determining the mean length and proportion of a given year class.

THEORY

It is a common assumption that each year class in a length distribution of fish has a normal length distribution and we will assume that also. We state this formally using an a for the age, using p(a) for the proportion of fish of age a, u(a) to be its mean and v(a) the variance.

$$(1) \quad P(x) = \sum_{a=1}^{N} p(a)Q\left((x-u(a))/\sqrt{v(a)} \right)$$

Where P(x) is the probability density function of the number of the number of fishes of length x. $\varphi(x)$ is the normal-function having the mean zero and variance 1.

The characteristic function for this distribution is easily obtained by finding the Fourier-transform of eq.(1) and is equal to: (The Fourier-transform of a normal function is a normal function.)

$$(2) \quad \bar{P}(w) = \sum_{a=1}^{N} p(a)\ \mathrm{Exp}(-w^2\ v(a)/2)\ \mathrm{Exp}(\ i\ u(a)\ w)$$

Where w is the frequency, i is the square-root of -1, and the bar over P denotes the "Fourier-transform of:".
Equation (2) can also be written as:

$$(3) \quad \bar{P}(w)\ \mathrm{Exp}(w^2\ v/2) = \sum_{a=1}^{N} p(a)\ \mathrm{Exp}(w^2\ (-(v(a)-v)/2)\ \mathrm{Exp}(iu(a)w) = \bar{P}_{-v}(w)$$

What we now have, on the right hand side of the equation, is the characteristic function of a length distribution that has year classes with the same means and proportions as the original distribution but has reduced variances and thus less overlapping. The variances are reduced by v, which is some arbitrary amount but must be less than the smallest variance in the original distribution.

Using a star "*" to denote a convolution we have:

$$(4)\ P(x) = \varphi(x/\sqrt{v}) * \sum_{a=1}^{N} p(a)\ \varphi(\ (x-u(a)/\sqrt{v(a)-v})$$

Or:

(5) $P(x) = \varphi(x/\sqrt{v}) * P_{-v}(x)$

To reduce the variances of all the year classes by v, equation (5) needs to be solved. (Equations of this kind are not totally unknown to fishery scientists. The "Craig and Forbes" equations are of the same kind with the target strength distribution playing the role of P_{-v} and the "directivity" distribution playing the role of $\varphi(x/\sqrt{v})$, ref (1).)

It is important to note that even if the year classes are not normally distributed, there is a good chance that eq.(5) holds for not too big a reduction in variance.
Each year class does not need to be normally distributed but only to consist of a distribution that may be obtained through a convolution of a normal-distribution with another distribution.
For instance it is likely that the error resulting from inaccurate measurement of each fish contributes to such a convolution.

To solve equation (5) it would be best to use all available information such as the known fact that $P_{-v}(x)$ can not be negative and all the available information on the growth of the fish species at hand should ideally be used.

Nevertheless we will here only solve (5) using straight-forward and fast methods.

It is obvious that it would be simple to calculate the numerical Fourier-transform of a measured length distribution to obtain the characteristic function of the distribution and then to divide that function by $Exp(-w^2v/2)$. The inverse Fourier-transform

should then yield the same distribution with reduced variances. (This is totally anlogous to the way the Craig and Forbes equations are usually solved, even if the deconvolution is usually done in the "time-domain")
The problem with this is that we don't know the exact length distribution, but only a sample from it, and the usual method of grouping the fishes into intervals results in a distribution that is certainly not composed of a sum of normal distributions only. (Because of inaccuracies).
The value of the length-distribution in each interval is only known with limited accuracy .
How to handle this inaccuary is the biggest problem in solving eq. (5), the method that we have chosen will be discussed in the next chapter.

The Practical Algorithm

The measured length distribution may be thought of as beeing composed of two distributions, one is the correct length distribution (divided into intervals) and the other is an error function ,and if we tried to deconvolute the measured length distribution using the above described method, this error function would dominate the output.
To increase accuracy in each interval the usual method among fishery scientists is to make them wider and fewer, but since this has the effect of increasing the variances of each year class, (which is opposite to what we want), we don't do that. Also, this may introduce what is known in digital signal processing text-books as "aliasing".
If the square-root is taken of the measured length distribution, then it is a good approximation (if the number of fishes in the interval are 10 or more) to assume that the error in each interval is constant with std.dev. of size 1/2. After taking the square-root we are left with two functions added together, one is the square-

root of the length distribution and the other is an uncorrelated error function, containing no information on the size distribution. We will try to remove as much of this error function as possible without harming the square-root of the length-distribution. We do this by calculating the Fourier-transform. The error function has equal chances of having power in all frequencies and we can calculate the probable sizes of this power, using the fact that the sum of squares for $E(x)$ and $\bar{E}(w)$ is proportional, where E is the error function, (the Parseval theorem).
Then we put the power of all frequencies to zero if it is not appreciably higher than what could be expected from the error function. The square-root of the length distribution is a low frequency function with its power rapidly diminishing with higher frequencies and the error function has equal power at all frequencies. So what we really do, is to low-pass filter the data. We only have to find the frequency for which the power in all higher frequencies could have originated by chance, and set this power equal to zero.
After the measured square-root of the measured length distribution has been cleaned in this way it is squared to obtain a smoothed approximation to the real length distribution. This function could then be Fourier-transformed and deconvoluted with a normal distribution having a variance v, to reduce the overlapping between year-classes.

All our numerical problems are not solved by the above described smoothing algorithm. Numerical error will seep in again when the the function is squared and the Fourier transform is taken, so instead of dividing the Fourier transform with $Exp(-v\ w^2/2)$ we add a small constant and divide with $Exp(-v\ w^2/2)+Exp(-v\ m^2/2)$ where m is the highest frequency that was let through when we smoothed our data. Frequencies above m are not worth much, and this will make sure that they are not excessively amplified. This constant has usually a very low value and will change little but for the frequencies above

m. (When the function is squared , power is invoked again in frequencies between m and 2m, this does not add real resolution however).

This last filtering may seem to be too arbitrary to many, a method that would seem more justifiable would be to put all power above 2m to zero, since theoritically there should be none. But experimentally this approach has not yielded as nice outputs as has this one. In our programs there is an option on which approach to use.

If v is well beneath the largest variance in the distribution there is little difference in these two approaches.

This algorithm may seem complicated , which is because the problem of getting rid of the error is complicated. Really this is a very simple approach, we employ it because it is non-iterative, numerically easy and seems to give acceptable results. It is best demonstrated through examples which will be given in the next chapter.

(We have tried to avoid messing up this paper by mentioning number of things that could improve this algorithm, such as using "windowing", and using iterative processes.)

We sum up the method as follows:

1 Group the fishes into length intervals, preferably not too wide (see 4) and so that at least 10 fishes are in each interval on the average.

2 Take the square-root of the resulting distribution.

3 Calculate the Fourier-transform of the distribution in 2.

4 Find the frequency, m, above which all power in the Fourier transform could have resulted from the inaccuracy in the

measured length distribution, and put all power above it to zero. (If the intervals have been choosen too wide this frequency cannot be found and aliasing has been introduced). We have done this visually, an automatic method has not yet been implemented into the programs.

5 Calculate the inverse Fourier-transform.

6 Square it. (The resulting distribution is our smoothed version of the length distribution).

7 Calculate the Fourier-transform.

8 Divide it with $\mathrm{Exp}(-w^2 v/2)+\mathrm{Exp}(-m^2 v/2)$, where v is the reduction in the variances of all the year classes that is sought for.

9 Calculate the inverse Fourier transform. What we now have is the distribution with reduced variances.

As can be seen there are a number of Fourier-transform calculations. These are straight-forward to do, using some of the Fast-Fourier-transform algorithms in the litterature, but since we prefer to use Basic as a programming language and very few good FFTs exist for that languge, we have written our own FFT that optimizes the number of additions as well as multiplications and is therefore especially fast running interpreter BASIC. There should be no machine on which the above algorithm would run too slowly. (A 64-point FFT costs only ~~768~~ 94 real multiplications and ~~1152~~ 420 real additions). A listing of our computer program is given in the appendices at the end of this paper.

EXAMPLES AND DEMONSTRATIONS

To demonstrate and test the above algorithm we begin by simulating a length distribution that could have been obtained for shrimp growing up at Iceland's North-West coast (in march). We use the following parameters. (table 1) (Values chosen by Unnur Skuladottir)

Table 1

Year class	Mean-length (mm)	Standard-deviation	Proportions
1	11.6	.9	21
2	15.3	1.0	34
3	18.4	1.05	25
4	21.0	1.1	15
5+	22.6	1.1	5

This distribution is drawn in figure 1.
We choose to sample 1500 shrimps randomly out of this distribution and to group them in half-millimeter groups. (This is common practice at our institute).
The sampled distribution is shown in fig.2.
The square-root of it is shown in fig.3. This function is composed of the square-root of the length distribution plus som random error that is very nearly uncorrelated and not a function of the length distribution. The error has a standard devitation equal to 1/2 in each interval.

The size of the Fourier-transform of the function in fig.3 is

shown in fig.4. This is the Fourier-transform (DFT) of the square-root of the length distribution which is a low-frequency function plus the Fourier-transform of the error function. The error function is equally likely to have power in all frequencies, so all the power in higher frequencies belongs to the error function.
We have drawn on the figure a line representing the mean value of the power of the error frequncies. (This value is calculated before-hand using the Parseval theorem).
It is obvious that all power above 11 could be accounted for by the error function and therefore contains no information on the length distribution, this power is therefore put to zero and the inverse Fourier transform is calculated and displayed in figure 5. As can be seen there are negative values in fig.5 which we put to o.

The function in figure 5 is squared and the result is shown in fig.6. This is then, the smoothed version of our sampled distribution. Note the similarity with fig. 1. Even if fig.1 and fig.6 seem alike there is much more resolution in fig.1 and there may not be enough resolution left in fig 4. to distinguish between all the year classes. (The resolution is reduced by 12/32 since only the 12 lowest frequencies were kept when smoothing and a 64 point DFT was used, which gives 32 useful frequencies.)
Next the Fourier transform is taken of the distribution in fig 6. It is divided by $\mathrm{Exp}(-v\, w^2/2)$ and the inverse transform taken (v is here set to 0.81, to reduce the variances by 0.81). The result is shown in fig. 7. And it is seen that the first four year classes stand out clearly with the correct mean lengths, but the fifth year class can barely be imagined. This is a great improvement over fig.2 from which it is derived.

In appendix B we do this twice again using the same data as in table 1. but using other random samples from the distribution, to show that fig. 7 is no coincidence.

We'll now demonstrate using real samples from shrimps caught in "Isafjardardjup", the Marine Research Institute has collected from this fjord in North-Western Iceland a large amount of data on a month to month basis for the last 10 years. Figure 8 shows us the length distribution of 5084 shrimps caught in February 1984, they are grouped in .5 mm intervals. The first peak belongs to the 19-82 year class. (We do not display a few shrimps belonging to the to the 1983 year class).

The size of the Fourier transform of the square-root is displayed in fig. 9. It is seen that only the 14 lowest frequencies have power that is appreciably higher than the error.

Figure 10 is the smoothed distribution deconvoluted so as to lessen the variances by .7 mm^2. It is seen that four peaks stand out and we have marked them with the name of the year classes that we belief they belong to. It is also seen that there is a false peak just before the 1982 year class, which is because this year class has a variance that is close to .7mm^2.

Real confidence in what we see in such distributions is only obtainable by looking at a series of the distributions in time. In doing this the human eye outperforms any computer program in detecting trends and patterns. So let's look at the distributions a month earlier and one year earlier.

Figure 11 is the distribution of 1705 shrimps from this same fjord in January 84, and fig.12 is what it looks like after our analysis. Figure 13 is the distribution of 6844 shrimps from the previous year February 1983, and fig 14 is what it looks like after similar analysis.

In general one can expect more resolution when the number of shrimps is increased. But often greater number is obtained by lenghtening the sample period and increasing the sea area, which then contributes to increased overlapping between year classes. (The shrimps grow while one is sampling and there may be added other growth stocks of shrimps by increasing the area.)

Shrimps are an important example to look at since there is no other known way for age determination of them other than analysis of their size distribution. We will now look at commercially a more important species.

Fig 15,17,19,21 and 23 are length distributions of cod collected by the Marine Res. Institute of Icel. off the North and East coast of Iceland ref (2). Figures 16,18,20,22 and 24 show those same length-distributions when they have been nonlinearly filtered (10-14 lowest frequencies were kept, we used 128-point discr. Four. tr.), and deconvoluted using $v \simeq 10$. More deconvolution would have caused false peaks when the first year-classes which have the least variations in lengths had split up. We have not done anything to the youngest year classes if those were already non-overlapping with the rest of the length-distribution.

The question arises whether one can have any faith in peaks that stand out after mathematical manipulations like these. The answer is no. Only if there is a basis for a peak in the original distribution, will a peak in the deduced distribution be worth anything. It is our belief that the eye is a very clever signal processor and it should be used together with common sense to inspect the original distribution to verify whatever has been deduced.

We have marked all the year classes with their age, we would not have been able to do this but for the first three years with any confidence, had we not had age-determinations from otoliths for comparision.

There are many interesting features in these length distributions, such as that the three year old cod gets smaller between February and May (this was also seen in the otoliths) and that almost all its growth is in May to August, and an especially good growth too, the first and second year grows more evenly. It is interesting to see how the proportions of year classes varies but this is not to be the subject of this paper. We will nevertheless add, that in other

years the young cods seem to behave differently.
What we want to emphasize is that one can get somewhat a feel of seeing what is happening by inspection of the length distributions and even if this often confronts one with puzzles there is a false security in just getting one figure for the proportion of a year class caught in one year and its mean length.
We end these demonstrations by displaying two length distributions for haddock. Fig. 25 shows the length distribution of 3408 haddocks caught off the North-Western coast of Iceland between 23. of February and 8. of March 1984. Fig. 26 shows what this distribution looks like after our analysis, (variances have been reduced by 6 cm^2, the first year class was left untouched and only the 16 lowest frequencys had power appreciably higher than the error). We have marked the peaks with their appropriate age, those were compared to otolithic age determination and found to be resonable.
Fig. 27 shows the distribution of 2229 haddocks caught in the same survey off the eastern coast of Iceland only half a month later.
Fig. 28 shows this distribution after similar analysis, and again we have marked the peaks with their appropriate year class. Otoliths were also collected for these haddocks and the peaks correspond nicely to the mean lengths of the year classes. It is striking how dissimilar the mean lengths are for those two areas, and it is clear that great care has to be taken when putting together age-length keys for use on commercially caught haddock from the otoliths collected in this survey.

CONCLUSIONS

We have pointed out that it should be possible to decrease overlapping between year classes in fish-length distributions by deconvoluting them with a normal function. We have also stated that there are practical numerical problems in doing so, and we have demonstrated a simple approach for solving those. This problem of forming a practical method for deconvoluting functions given restrictions, such as that the output should be positive and given the properties of the error, are well known in other sciences. Because of our incomplete knowledege of this field we are nevertheless not able to give any references to good appropriate methods, but we must say that attempts have been made to solve equations of the same kind as eq. (5), using far more sophisticated methods than what we have used.

As a first attempt of bringing signal processing into fishery sciences, the results in this paper seem to be promising.

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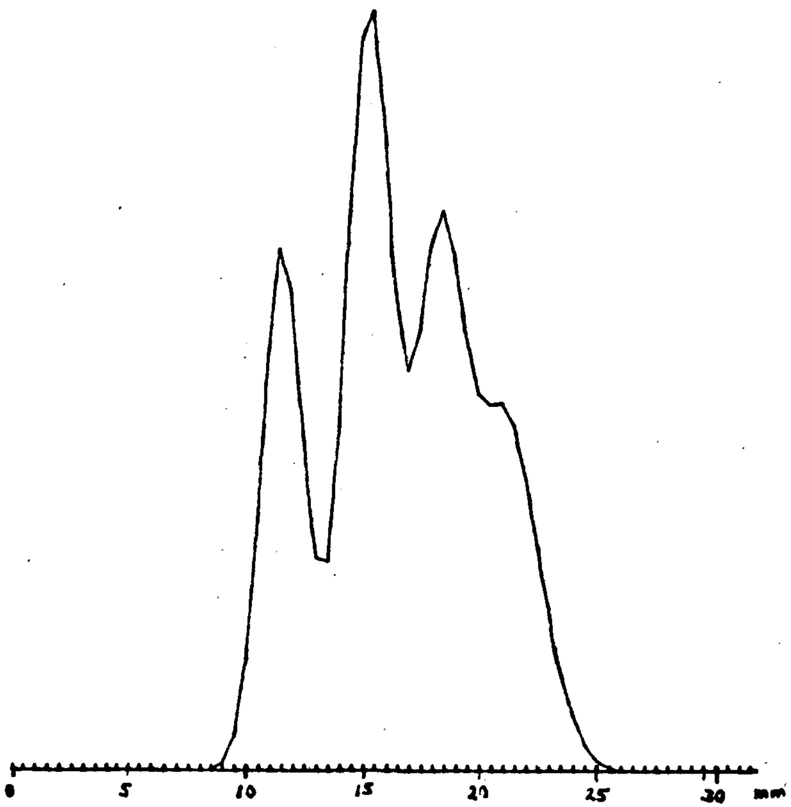

Figure 1

The true distribution using the values in table 1.

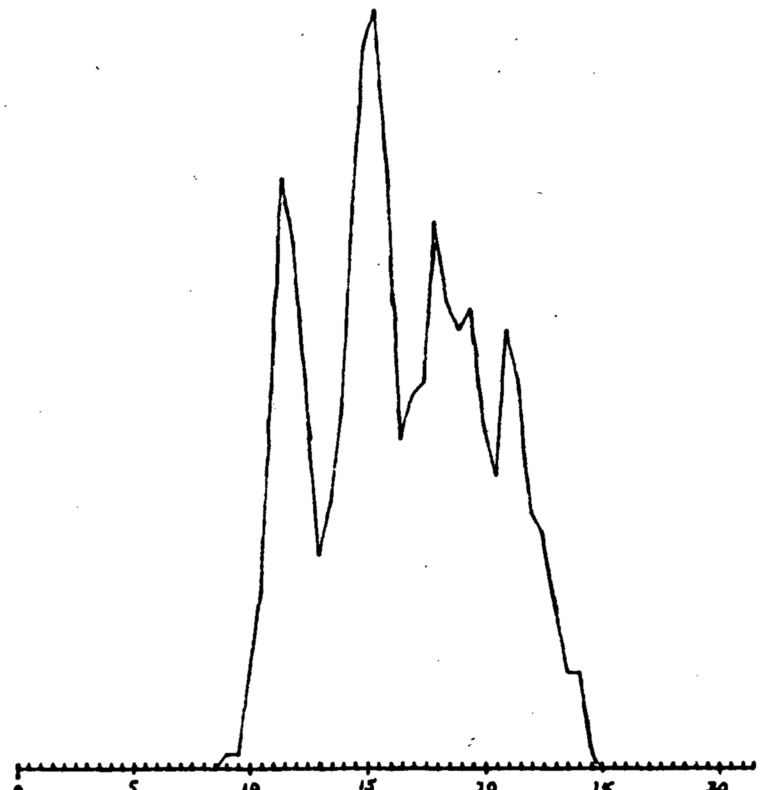

Figure 2

The distribution of 15oo "shrimps" sampled out of the distribution in fig. 1.

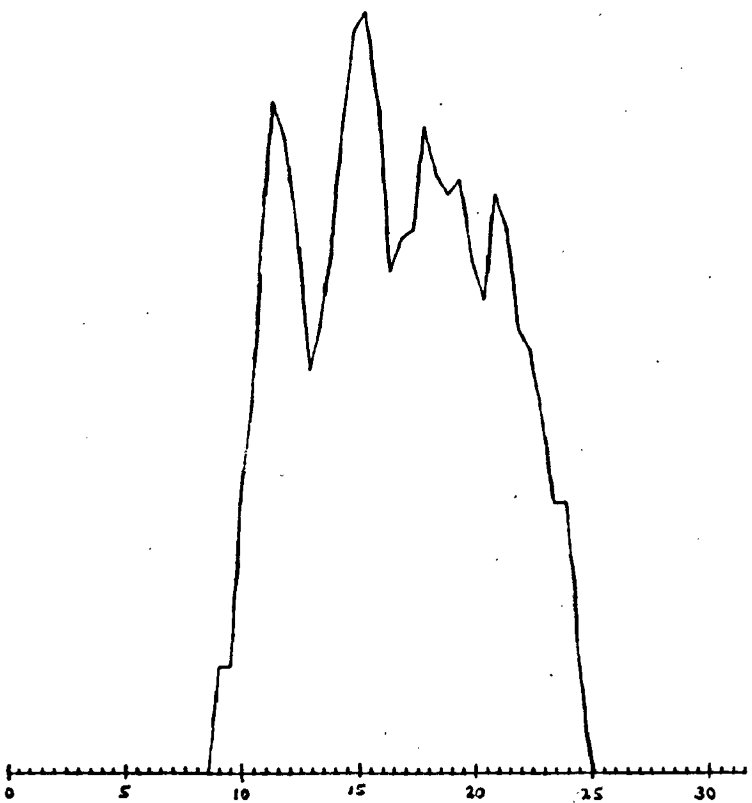

Figure 3
The square-root of the sampled distribution.

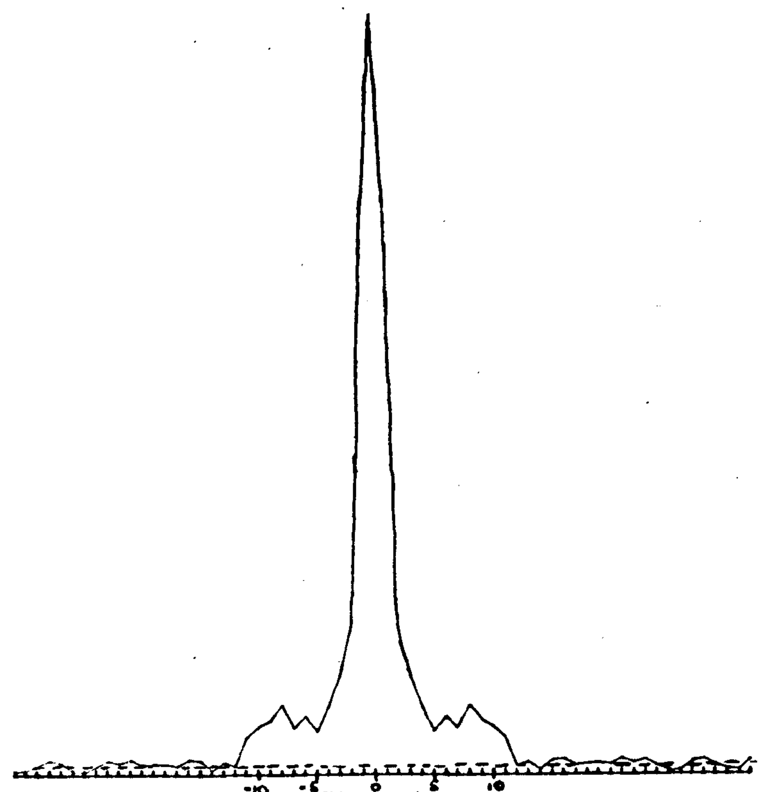

Figure 4
The size of the 64 point DFT of the function in fig.3.

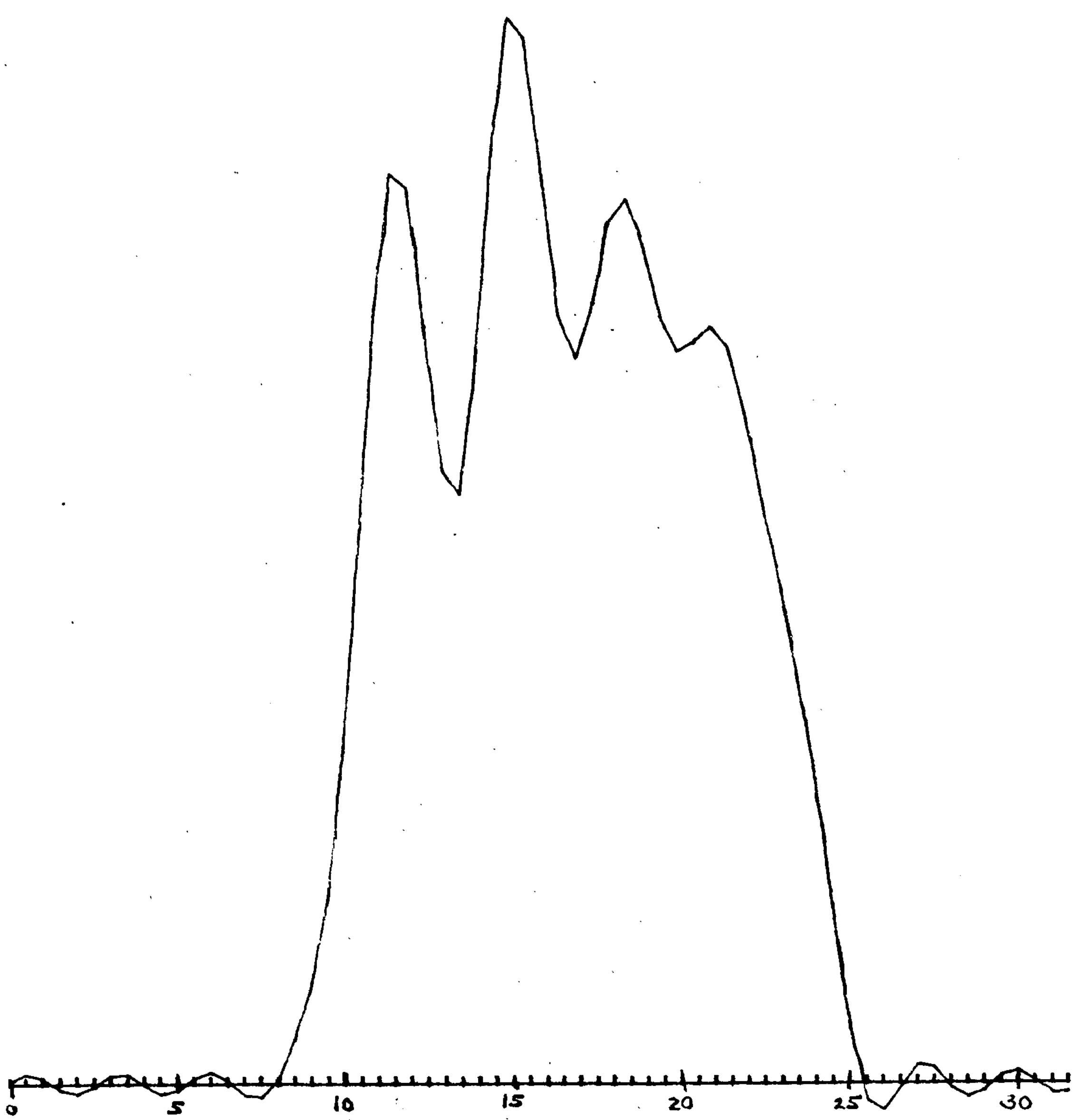

Fig.5

The inverse tranform of fig.4 (High freq.power set to 0)

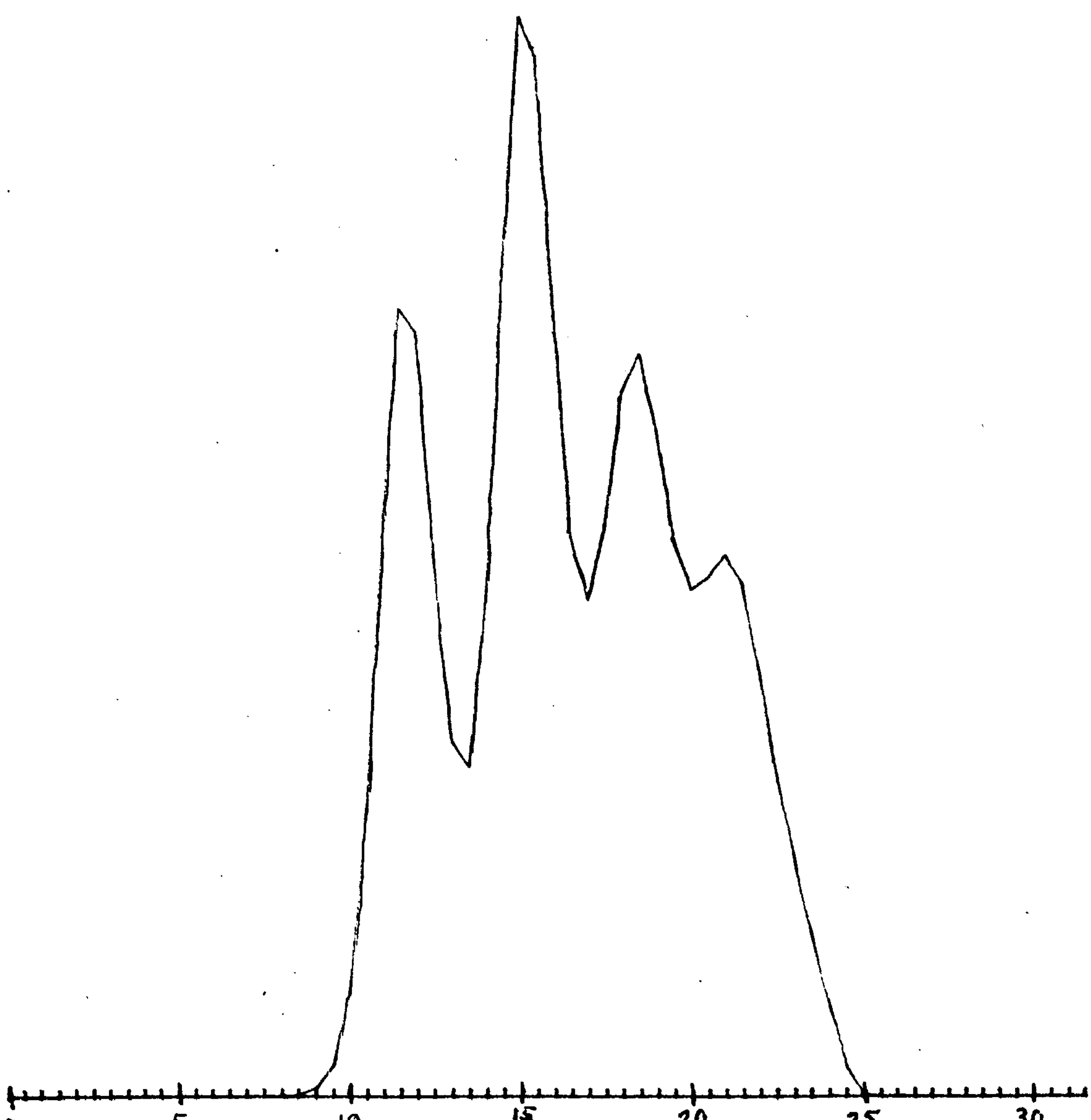

Fig.6
The function in fig.5 squared.

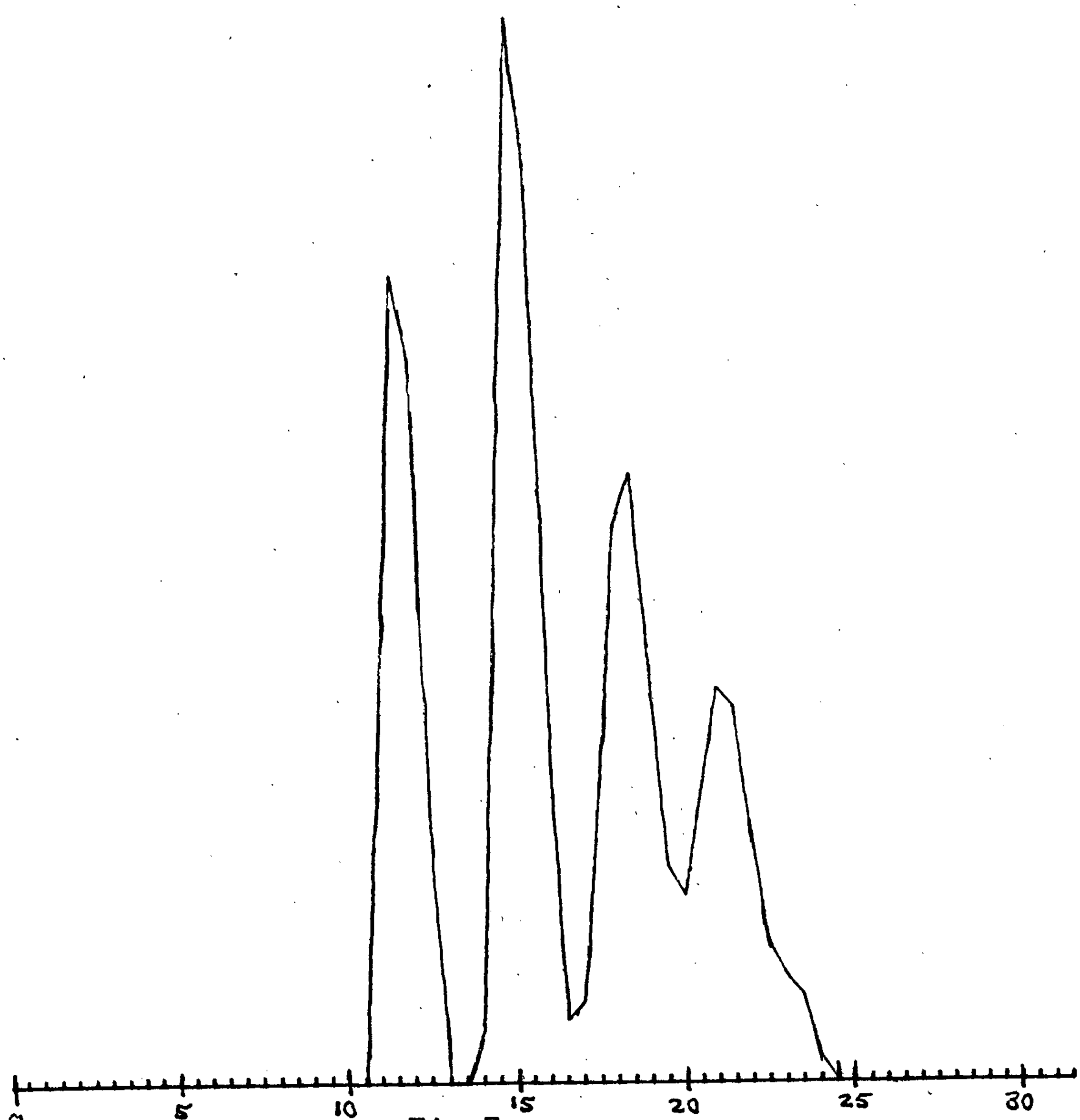

Fig.7

The function in fig.6 deconvoluted.
(negative values put to 0)

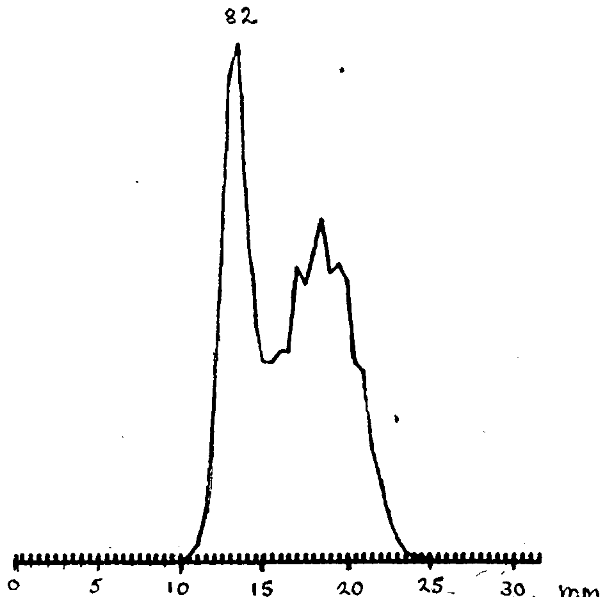

Fig.8 5084 shrimps in Feb.84

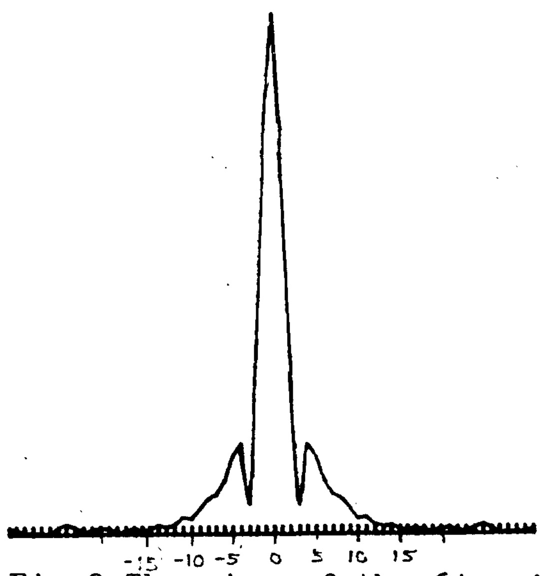

Fig.9 The size of the 64-point FFT

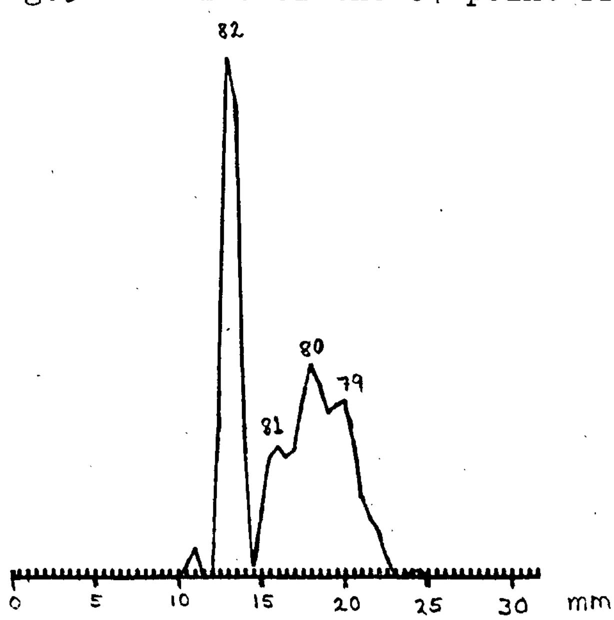

Fig.10 After reducing the var. by .7

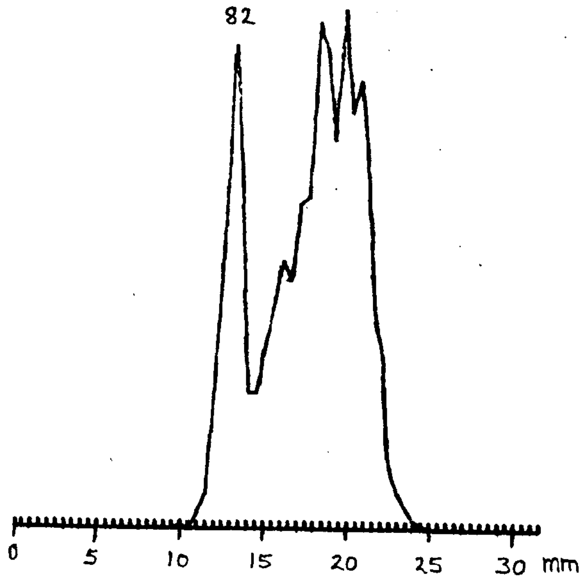

Fig.11
1705 shrimps a month earlier.

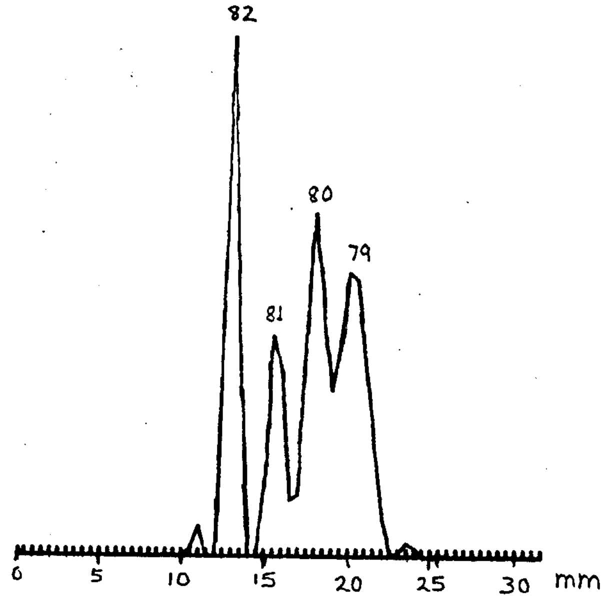

Fig.12
After deconvoluting.

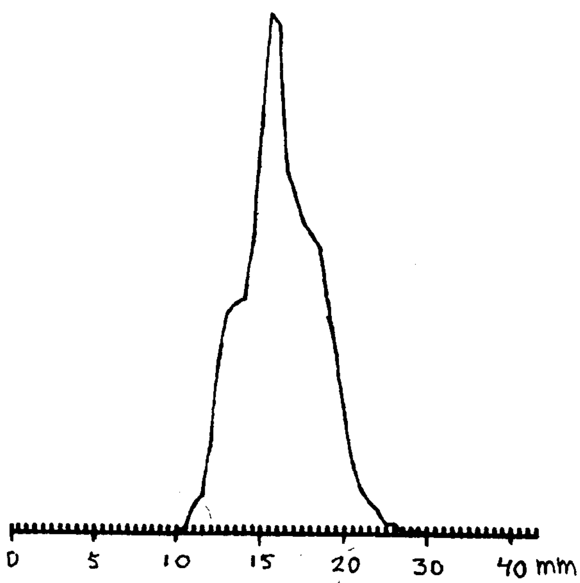

Fig.13
6844 shrimps a year earlier.

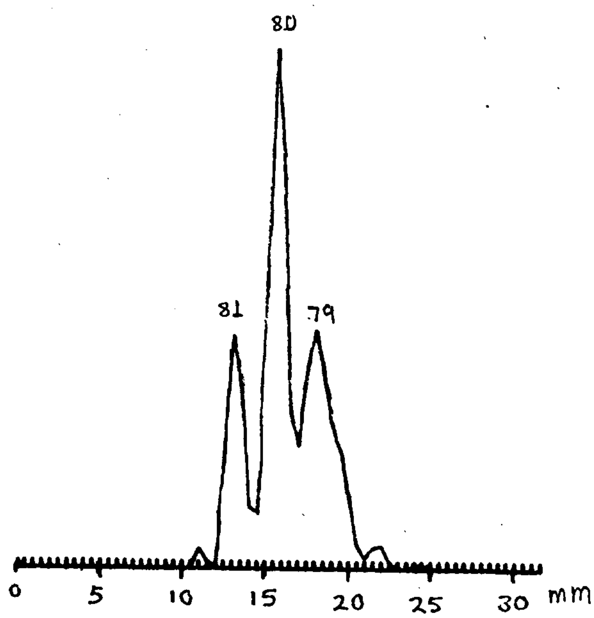

Fig.14
After deconvoluting.

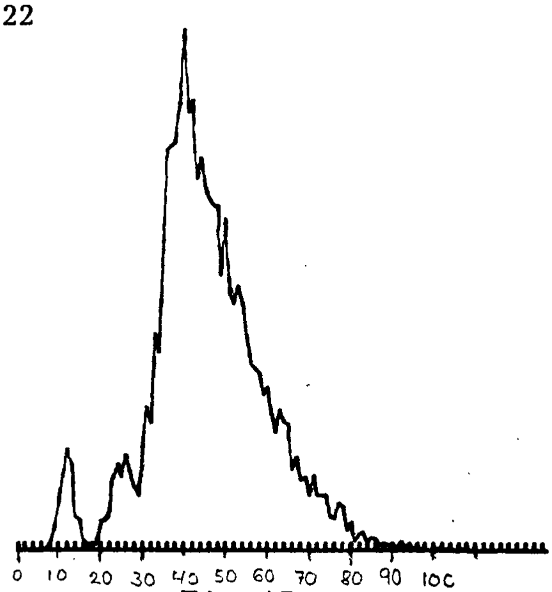

Fig.15
6080 cods in Feb.76

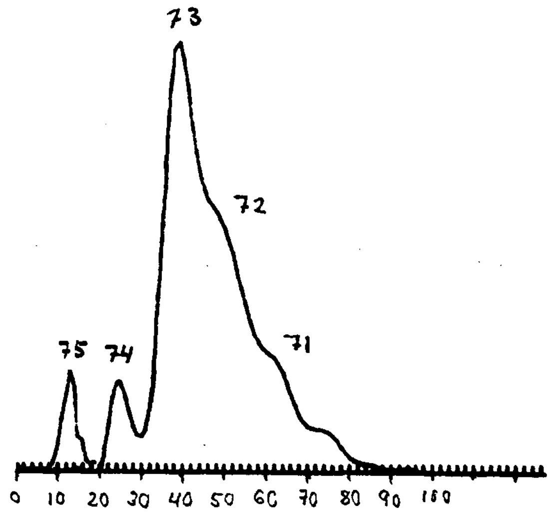

Fig.16

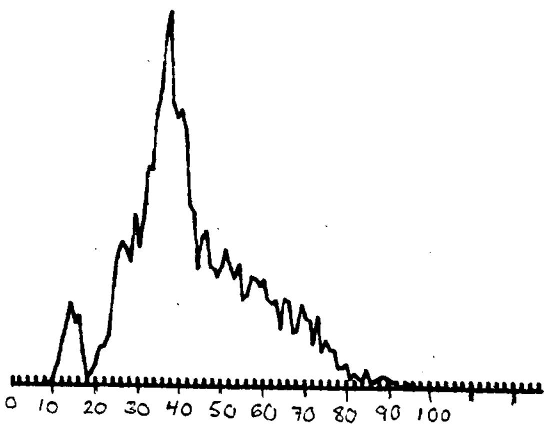

Fig.17
4268 cods in May 1976

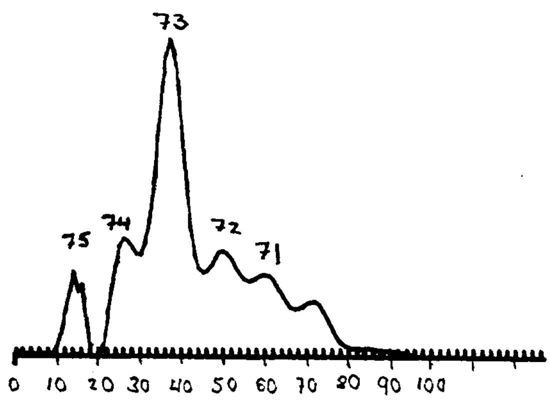

Fig.18

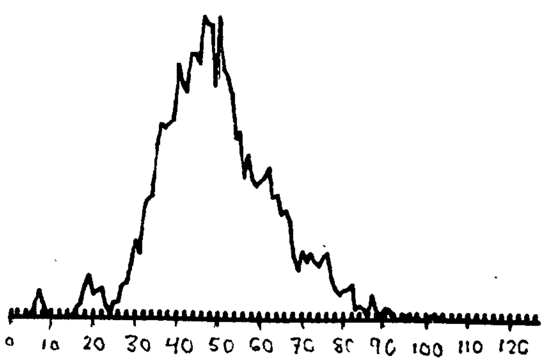

Fig.19
4475 cods in Aug. 1976

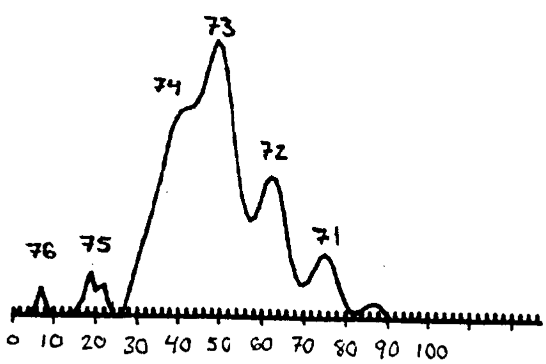

Fig.20

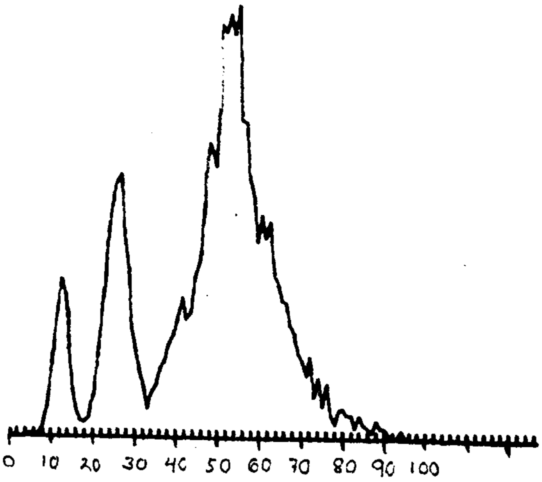

Fig.21
5524 cods in March 1977

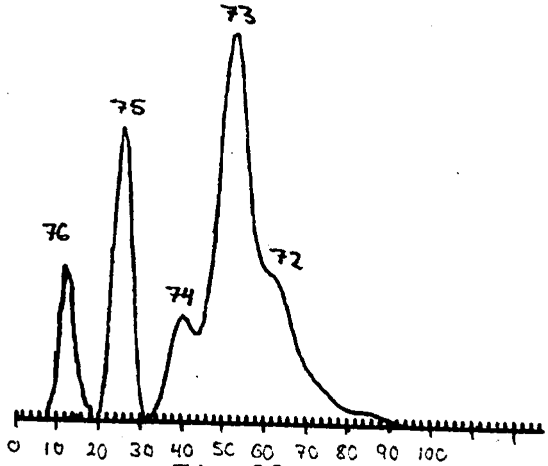

Fig.22

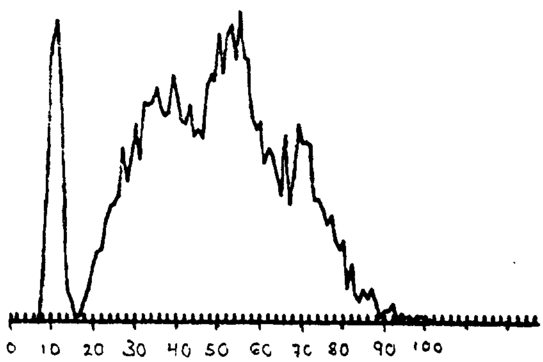

Fig.23
6332 cods in March 1979

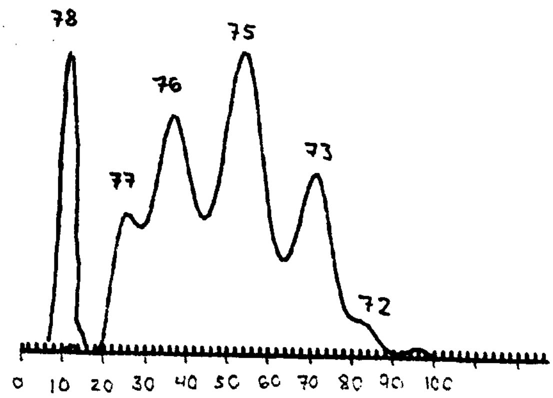

Fig.24

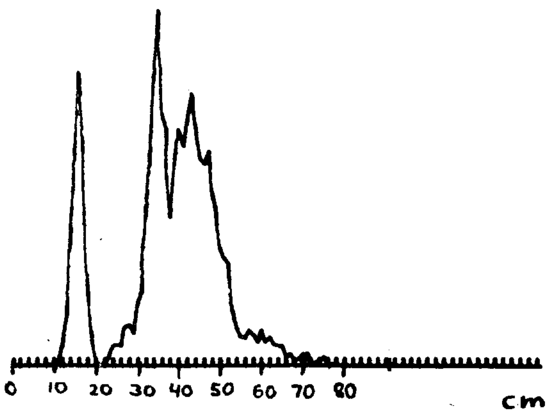

Fig.25
3408 haddocks north-west off Iceland.

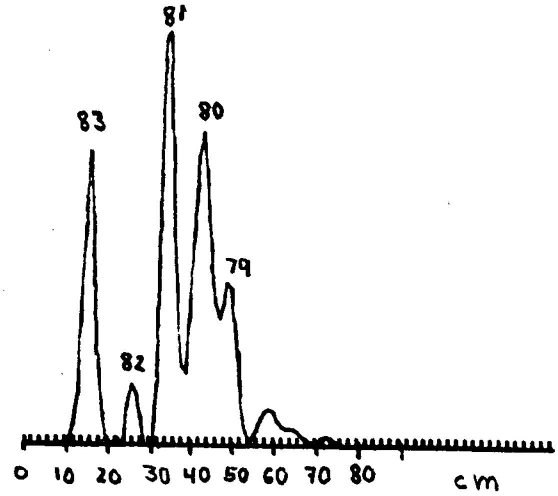

Fig.26

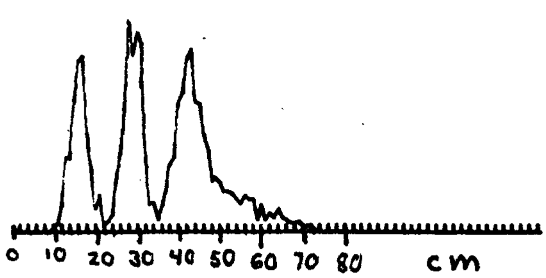

Fig.27
2229 haddocks east off Iceland.

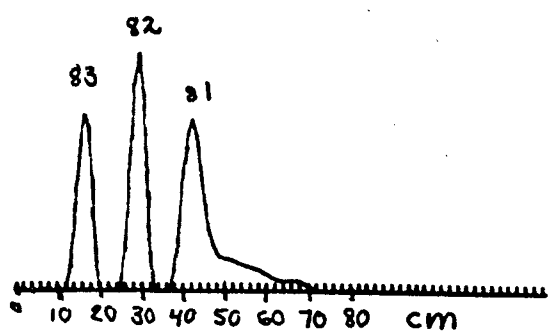

Fig.28

Appendix A

```
10 DIM D(128),S(16),S1(16),S2(16)
20 PRINT "  An interactive program for smoothing and sharpening"
30 PRINT "  size distributions"
40 PRINT "    Copyright 84 Marine Research Institute of Iceland" \ PRINT
50 PRINT "Enter a number that is a power of two and is larger"
60 PRINT "than the number of entries in the fish-distribution"
70 INPUT N
80 Q2=1/SQR(2)
90 GOSUB 1450 \ REM Calculate the multiplercoefficients
100 PRINT "                    MENU" \ PRINT
110 PRINT " 0          DISPLAY MENU"
120 IF I8=0 THEN PRINT " 1             INPUT DATA"
130 IF I8=0 THEN PRINT " 2             SQUARE-ROOT"
140 IF I8=0 THEN PRINT " 3             FAST FOURIER TRANSFORM"
150 IF I8=1 THEN PRINT " 4             PUT HIGH FREQUENCY POWER TO NULL"
160 IF I8=1 THEN PRINT " 5             INVERSE FFT"
170 IF I8=0 THEN PRINT " 6             PUT NEGATIVE VALUES TO NULL"
180 IF I8=0 THEN PRINT " 7             SQUARE DATA"
190 PRINT " 8          DISPLAY DATA"
200 IF I8=1 THEN PRINT " 9             DIVIDE EXP(-wwv)+EXP(-wwm)"
210 IF I8=1 THEN PRINT " 10            DIVIDE EXP(-wwv)"
220 PRINT " 11         OUTPUT DATA"
230 PRINT " 12         STOP" \ PRINT
240 PRINT "ENTER OPTION NUMBER" \ INPUT C \ C=C+1
250 ON C GOSUB 100,280,430,850,450,770,510,530,1510,550,620,690,720,3000
260 PRINT "DONE"
270 GO TO 240
280 PRINT "Enter name of raw data file"
290 INPUT A$ \ OPEN A$ FOR INPUT AS FILE #1
300 PRINT "How many entries are there in that file" \ INPUT N3
310 PRINT "What is the width of each interval" \ INPUT B
320 T5=0 \ I8=0
330 FOR I=0 TO N3-1 \ INPUT #1,D(I) \ T5=T5+D(I)
340 PRINT D(I),I*B \ NEXT I
350 PRINT "First non-zero interval"; \ INPUT M5
360 FOR I=0 TO M5/B-1 \ D(I)=0 \ NEXT I
370 PRINT "Last non-zero interval"; \ INPUT M6 \ E=SQR((M6-M5)/B)/2
380 FOR I=M6/B+1 TO N-1 \ D(I)=0 \ NEXT I
390 PRINT "Number of measured fishes;",T5
400 CLOSE #1
410 FOR I=N3 TO N-1 \ D(I)=0 \ NEXT I
420 RETURN
430 FOR I=0 TO N-1 \ D(I)=SQR(D(I)) \ NEXT I
440 RETURN
450 PRINT " IF SQR HAS BEEN TAKEN THE STD.DEV OF THE NOISE IS ";
451 PRINT E
452 PRINT " OTHERWISE IT IS EQUAL TO THE SQR OF THE NUMBER OF FISHES";
453 PRINT SQR(T5) \ PRINT
454 PRINT " USE THIS INFORMATION TO DETERMINE WHEN IT CAN NO LONGER"
455 PRINT " BE ASSUMED THAT THE POWER BELONGS TO THE ERROR FUNCTION"
456 PRINT " -- NOTE THAT IF THE SQUARE-ROOT HAS BEEN TAKEN, AND IF"
457 PRINT " THE NUMBER OF NON-ZERO INTERVALS IS K, AND A N-POINT FFT "
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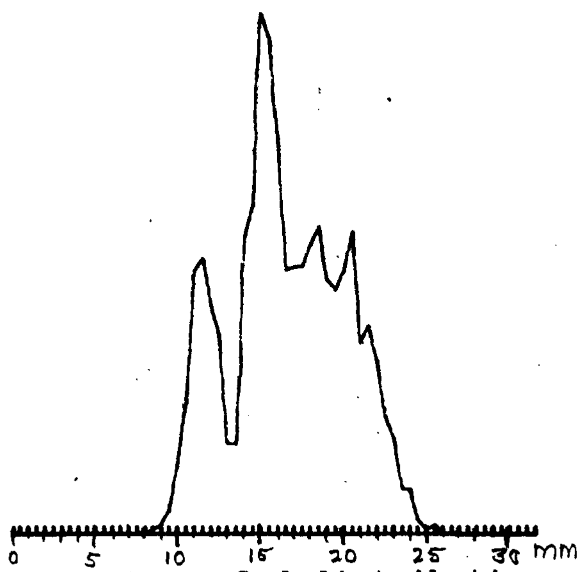
```
458 PRINT " HAS BEEN TAKEN THEN EVERY K/N VALUE IS UNCORRELATED.  "
459 PRINT " THE REST MAY BE THOUGHT OF AS BEEING INTERPOLATED. "
460 PRINT " BEWARE OF DIGIT PREFERENCES !" \ PRINT
461 PRINT "REAL PART","IMAG","FREQ"
462 FOR I=N/2 TO 0 STEP -1 \ PRINT D(I),D(N-I),I \ NEXT I
470 PRINT "WHAT IS THE HIGHEST FREQUENCY THAT YOU WANT TO USE"
480 INPUT M6
490 FOR I=M6+1 TO N/2 \ D(I)=0 \ D(N-I)=0 \ NEXT I
500 RETURN
510 FOR I=0 TO N-1
512 IF D(I)<0 THEN D(I)=0
516 NEXT I
520 RETURN
530 FOR I=0 TO N-1 \ D(I)=D(I)^2 \ NEXT I
540 RETURN
550 PRINT "DECREASE IN VARIANCE" \ INPUT V
560 S=SQR(V)/B \ O=EXP(-2*(S*PI*M6/N)^2)
570 FOR I=0 TO N/2-1 \ X=EXP(-2*(S*PI*I/N)^2)
580 D(N-I)=D(N-I)/(X+O) \ D(I)=D(I)/(X+O)
590 NEXT I
600 X=EXP(-2*(S*PI/2)^2) \ D(N/2)=D(N/2)/(X+O)
610 RETURN
620 PRINT "DECREASE IN VARIANCE" \ INPUT V
630 S=SQR(V)/B
640 FOR I=0 TO 2*M6 \ X=EXP(-2*(S*PI*I/N)^2)
650 D(N-I)=D(N-I)/X \ D(I)=D(I)/X
660 NEXT I
680 RETURN
690 PRINT "ENTER A FILENAME" \ INPUT A$ \ OPEN A$ FOR OUTPUT AS FILE #1
700 FOR I=0 TO N-1 \ PRINT #1,D(I) \ NEXT I \ CLOSE #1
710 RETURN
720 STOP
730 REM  This program calculates the inverse transform of real sequence.
740 REM  The real part is stored in D(i). The imaginary part in D(N-i),
750 REM  the same vecto.(Only half of them need to be stored). On output
760 REM  the real sequence is in this same vector.
770 FOR I=1 TO N/2-1 \ X=D(I) \ Y=D(N-I) \ D(I)=X-Y \ D(N-I)=X+Y \ NEXT I
780 GOSUB 850
790 FOR I=1 TO N/2-1 \ X=D(I)/N \ Y=D(N-I)/N \ D(I)=X-Y \ D(N-I)=X+Y \ NEXT I
800 D(N/2)=D(N/2)/N \ D(0)=D(0)/N
810 RETURN
820 REM  This is a FFT the input sequence ( must be real), is stored
830 REM  in D(i), on output the real part is in D(i) and the imag.
840 REM  in D(N-i). Only half of each needs to be stored.
850 GOSUB 1340
860 I8=1-I8 \ L=0
870 IF N=4 THEN 1220
880 N=N/2
890 GOSUB 870
900 L=L+N
910 GOSUB 1000
920 J=L+N \ K=L \ L=L-N
930 Y=D(K) \ X=D(L) \ D(K)=X-Y \ D(L)=X+Y
```

```
940 FOR I=1 TO N/2-1
950 X=D(L+I) \ Y=D(K-I) \ Z=D(K+I) \ W=D(J-I)
960 D(K-I)=X-Z \ D(L+I)=X+Z \ D(K+I)=W-Y \ D(J-I)=Y+W
970 NEXT I
980 N=N*2
990 RETURN
1000 IF N=4 THEN 1290
1010 N=N/2
1020 GOSUB 870
1030 L=L+N
1040 FOR I=0 TO N/2-1
1050 K=L+N-I-1 \ X=D(L+I) \ D(L+I)=D(K) \ D(K)=X
1060 NEXT I
1070 GOSUB 870
1080 L1=L+N \ K=L \ K1=L \ N2=N/2 \ L=L-N
1090 X=D(L) \ Y=D(K) \ Z=D(L+N2) \ W=D(K+N2)
1100 D(L)=X+Y \ D(L+N2)=(Z+W)*Q2 \ D(K)=X-Y \ D(K+N2)=(Z-W)*Q2
1110 M=N4/N \ H=M
1120 FOR I=L+1 TO L+N2-1
1130 S=S(H) \ S1=S1(H) \ S2=S2(H) \ H=H+M \ K=K+1 \ K1=K1-1 \ L1=L1-1
1140 R1=D(I)+D(K) \ R2=D(K1)-D(L1) \ C1=D(I)-D(K) \ C2=D(K1)+D(L1)
1150 G1=(R1-C2)*S \ G2=(C1-R2)*S \ F1=R1*S2 \ F2=C1*S2 \ H1=C2*S1 \ H2=R2*S1
1160 D(K1)=G1+H1 \ D(I)=G1+F1 \ D(K)=G2+F2 \ D(L1)=G2+H2
1170 NEXT I
1180 N=N*2
1190 RETURN
1200 REM
1210 REM This is a routine that calculates the transform of 4-points
1220 L1=L+1 \ L2=L+2 \ L3=L+3
1230 R=D(L)+D(L1) \ D(L1)=D(L)-D(L1)
1240 R2=D(L2)+D(L3) \ D(L3)=D(L2)-D(L3)
1250 D(L)=R+R2 \ D(L2)=R-R2
1260 RETURN
1270 REM
1280 REM This rouitine calculates the FFT of 4-point odd-sequence
1290 L1=L+1 \ L2=L+2 \ L3=L+3
1300 R=D(L) \ X=R+D(L1) \ Y=R-D(L1)
1310 R=D(L2) \ Z=R+D(L3) \ W=D(L3)-R
1320 D(L)=X+Z \ D(L1)=(Y+W)*Q2 \ D(L2)=X-Z \ D(L3)=(Y-W)*Q2
1330 RETURN
1340 REM  This is a bitreversing subroutine.
1350 L1=0
1360 FOR I=1 TO N-1
1370 N2=N
1380 N2=N2/2
1390 IF L1+N2>=N THEN 1380
1400 L1=L1-INT(L1/N2)*N2+N2
1410 IF L1<=I THEN 1430
1420 R=D(I) \ D(I)=D(L1) \ D(L1)=R
1430 NEXT I
1440 RETURN
1450 REM This routine calculates the multipliercoefficients
1460 N4=N/4
```

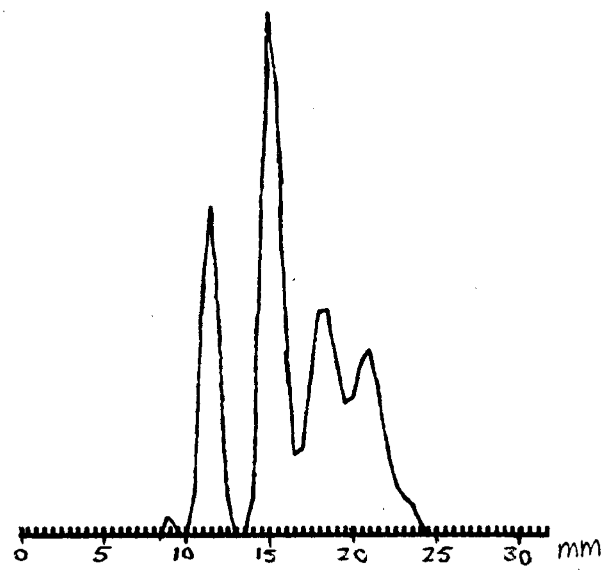
```
1470 R=2*PI/N
1480 FOR I=1 TO N/8 \ C=COS(I*R) \ S=SIN(I*R)
1490 S(I)=S \ S1(I)=C+S \ S2(I)=C-S \ NEXT I
1500 RETURN
1510 X=0 \ Y=0
1520 FOR I=0 TO N-1
1530 IF X<D(I) THEN X=D(I) \ IF Y>D(I) THEN Y=D(I)
1540 NEXT I
1550 FOR I=0 TO N-1
1560 FOR J=Y/(X-Y) TO D(I)*60/(X-Y) \ PRINT "X"; \ NEXT J
1570 PRINT I*B
1580 NEXT I
1590 RETURN
3000 S=0\FOR I=0 TO N-1\S=S+D(I)^2\NEXT I
3010 PRINT S*2-D(0)-D(N/2)
3030 RETURN
```

Appendix B

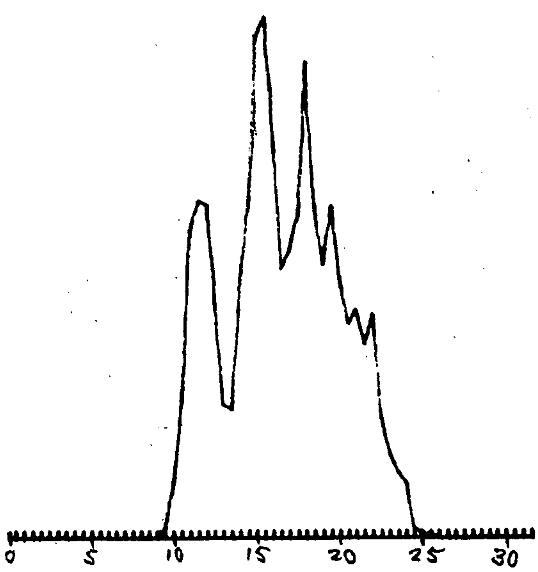
Two simulations more, using the values in table 1



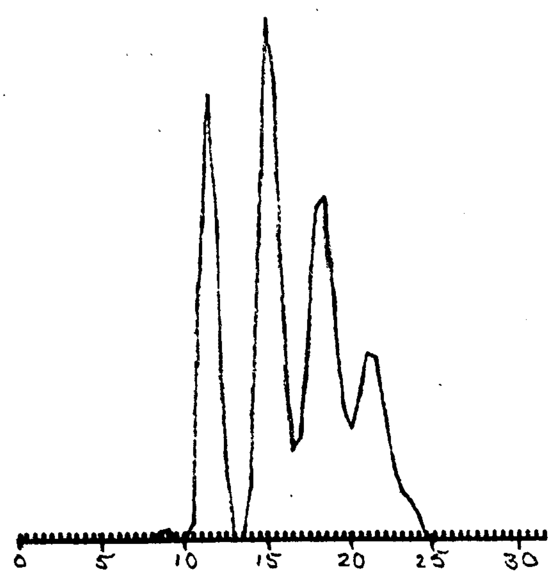
A sampled distribution



After the deconvolution



Another 1500 lengths sample



After the deconvolution